### ABSTRACT ALGEBRA QUALIFYING EXAM PROBLEM SESSION: EXTRA REVIEW

CALEB SPRINGER

Let me know if you find typos or have questions! Email: cks5320@psu.edu

These problems are intended for extra practice. Penn State grad students should complete the old exams first before starting extra problems: https://math.psu.edu/graduate/qualifying-exams

## 1. GROUP THEORY

**Problem 1** (University of Michigan May 2019). Suppose p > q are prime numbers, and that G is a group of order  $p^2q^2$ .

- (1) Show that p = 3 or G has a normal subgroup of order  $p^2$ .
- (2) If p = 3 (and therefore q = 2), show that G has a normal subgroup of order 3 or 9.

**Problem 2** (University of Michigan May 2018). Classify all groups (up to isomorphism) that have only one automorphism.

**Problem 3** (University of Michigan September 2018). Suppose that the finite group G acts transitively on the set X. For a prime p, let S be a Sylow p-subgroup of G and N(S) be its normalizer. Define

$$F = \{ x \in X \mid g \cdot x = x \ \forall g \in S \}$$

as the set of points fixed by S. Prove that N(S) leaves F invariant and acts transitively on F. Be sure to clearly indicate the parts of Sylow theory which you use.

**Problem 4** (University of Michigan May 2017). How many isomorphism classes of abelian groups of order  $6^4$  are there?

Date: November 22, 2020.

**Problem 5** (University of Michigan May 2017). Let G be a finite group and let p be a prime number. Show that the following conditions are equivalent.

- (a) The group G acts transitively on a set X such that the cardinality of X is at least 2 and relatively prime to p.
- (b) The order of G is not a power of p.

**Problem 6** (University of Michigan January 2016). Let p be a prime number and  $1 \le n < p^2$  an integer. Show that every p-Sylow subgroup of  $S_n$  is abelian.

**Problem 7** (University of Michigan September 2016). Suppose that  $H \subseteq \mathbb{Z}^2$  is the subgroup generated by (5,4) and (2,7). Show that there is a unique subgroup  $M \subseteq \mathbb{Z}^2$  of index 9 in  $\mathbb{Z}^2$  that contains H. Give explicit generators for M.

# 2. Ring Theory

**Problem 8** (University of Michigan May 2018). Let R be the ring

$$\mathbb{Z}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[2]{2^2} \mid a, b, c \in \mathbb{Z}\}$$

and let I = (5) be the ideal of R generated by 5. Write R/(5) as a product of fields.

**Problem 9** (University of Michigan September 2018). What are the prime ideals in the ring  $\mathbb{F}_3[x]/(x^5 + x^4 + 1)$ ? Which of these prime ideals are maximal?

Problem 10 (University of Michigan May 2016).

- (a) Suppose I is an ideal in a PID R such that  $I^2 = I$ . Show that I = (0) or I = R.
- (b) Give an example of an ideal I in a commutative ring R such that  $I^2 = I$  but I is neither (0) nor R.

**Problem 11** (University of Michigan September 2016). Suppose R is the subring of  $\mathbb{Z}[x]$  consisting of all polynomials  $f(x) = a_n x^n + \cdots + a_1 x + a_0$  where  $a_n, \ldots, a_1$  are all even (but  $a_0$  does not have to be even).

(a) Show that R contains a maximal ideal which is not finitely generated.

(b) What is the ring R/(3)? Is it finite?

### 3. Modules

**Problem 12** (University of Michigan September 2018).

- (a) Give an example of a nonzero finitely generated  $\mathbb{Z}[X]$ -module M which is torsion-free, but not free.
- (b) Give an example of a nonzero finitely generated  $\mathbb{Z}[X]$ -module M and two irreducible elements  $f_1, f_2 \in \mathbb{Z}[X]$  such that  $f_1f_2$  kills M, but M does not decompose into a product  $M_1 \times M_2$  such that  $f_1$  kills  $M_1$  and  $f_2$  kills  $M_2$ .

### 4. FIELD THEORY AND GALOIS THEORY

**Problem 13** (Boston College Spring 2018). Consider the following situation: E/F is a Galois extension of degree 4, and K/F is a non-Galois extension of degree 5 such that the compositum KE/F is Galois. Either prove such a situation is impossible or give an example of such F, E, K and prove that your example works.

**Problem 14** (University of Michigan May 2018). Suppose that F is a field,  $p(x) \in F[x]$  is a separable, irreducible polynomial of degree 3 with roots  $\alpha_1, \alpha_2, \alpha_3$ .

(1) Show that if the characteristic of F is not 2 or 3, then

$$F(\alpha_1, \alpha_2, \alpha_3) = F(\alpha_1 - \alpha_2).$$

(2) Show that if F has characteristic 3, then  $F(\alpha_1, \alpha_2, \alpha_3) \neq F(\alpha_1 - \alpha_2)$  is possible.

**Problem 15** (University of Michigan May 2017). Let  $\zeta_n = e^{2\pi i/n}$  be a primitive *n*-th root of unity.

(a) For which positive integers n does  $\mathbb{Q}(\zeta_n)$  contain  $\sqrt{2}$ ?

(b) For which positive integers n does  $\mathbb{Q}(\zeta_n)$  contain  $\sqrt[3]{2}$ ?

**Problem 16** (University of Michigan May 2016). Consider the polynomial  $p(x) = x^9 + 1 \in \mathbb{F}_2[x]$ .

- (a) Find the splitting field of p(x) over  $\mathbb{F}_2$ .
- (b) Show that  $p(x) = (x+1)(x^2 + x + 1)(x^6 + x^3 + 1)$  is the irreducible factorization<sup>1</sup> of p.

<sup>&</sup>lt;sup>1</sup>For the sake of this problem, you can take it as given that the lefthand and righthand sides are equal after multiplying the factors together. Just check that each factor on the righthand side is irreducible over  $\mathbb{F}_2$ .

(c) How many units does the ring  $\mathbb{F}_2[x]/(p(x))$  have?

**Problem 17** (University of Michigan September 2016). Let M be a field containing  $\mathbb{F}_p$ , and let K and L be subfields of M. Assume that the number of elements of K, L and M are  $p^6$ ,  $p^{10}$  and  $p^{60}$ , respectively. How many elements do the fields KL and  $K \cap L$  have?

**Problem 18** (University of Vermont January 2009). Let  $K_1$  and  $K_2$  be finite abelian Galois extensions of F contained in a fixed algebraic closure of F. Show that their composite  $K_1K_2$  is also a finite abelian Galois extension of F.

**Problem 19** (Kent State). Show that if F is a finite field, then every algebraic extension of F is separable.

**Problem 20** (Kent State). Show that if F is a finite field, then the multiplicative group  $F^{\times}$  is cyclic.

- **Problem 21** (Harvard Fall 2016). (a) Let F be a field and let  $G \subseteq F^{\times}$  be a finite multiplicative subgroup. Prove that G is cyclic.
- (b) Let  $F \supseteq \mathbb{Q}$  be a splitting field for the polynomial  $f = x^n 1$ . Show that  $F/\mathbb{Q}$  is a Galois extension with abelian Galois group.

**Problem 22** (Kent State). Let  $\mathbb{F}_{81}$  be the field of 81 elements.

- (a) Find all subfields of  $\mathbb{F}_{81}$ .
- (b) Determine the number of primitive elements for  $\mathbb{F}_{81}$  over the field  $\mathbb{F}_3$  of 3 elements, i.e. elements  $\alpha$  such that  $F_{81} = \mathbb{F}_3(\alpha)$ .
- (c) Find the number of generators for the multiplicative group  $\mathbb{F}_{81}^{\times}$ .

**Problem 23** (Kent State). Find the splitting field of  $f(x) = x^4 + x^3 + 4x - 1 \in \mathbb{F}_5[x]$ , and determine its Galois group.

**Problem 24** (Harvard Spring 2014). Let  $\mathbb{F}_q$  be a finite field with q elements. Count the number of monic irreducible polynomials of degree 12 over  $\mathbb{F}_q$ 

**Problem 25** (Rice Fall 2016). Let  $\mathbb{F}_7$  be the finite field with 7 elements, and let  $\beta$  be a cubed root of 2, i.e.  $\beta^3 = 2$ .

(a) Compute the minimal polynomial for  $\alpha = \beta + 1$  over  $\mathbb{F}_7$ .

(b) Determine, with proof, whether or not  $F(\alpha)$  is Galois over F.

**Problem 26** (Rice Fall 2016). Let K be a finite field with 64 elements. Describe all subfields of K.

4