ABSTRACT ALGEBRA QUALIFYING EXAM PROBLEM SESSION: FIELD AND GALOIS THEORY

CALEB SPRINGER

Let me know if you find typos or have questions!
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Unless noted otherwise, the questions come from:
The Pennsylvania State University's Abstract Algebra Qualifying Exam (2008-2020).
https://math.psu.edu/graduate/qualifying-exams

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1. Polynomials

Problem 1 (Fall 2008). *Let $\mathbb{F}_{11}$ be the field of 11 elements and let K be the splitting field of $x^3 - 1$ over $\mathbb{F}_{11}$. How many roots does $(x^2 - 3)(x^3 - 3)$ have in K?*

Problem 2 (Fall 2010). *Determine the degree of the irreducible factors of $x^n - 1$ over $\mathbb{Q}$ where $n = 3 \cdot 7 \cdot 11$.*

Problem 3 (Fall 2011). *Let K be a field of characteristic $p > 0$. Take a polynomial $f(X) = X^p - X - c$ in $K[X]$, and suppose that f has one root α in K. Prove that*

$$f(X) = (X - \alpha)(X - (\alpha + 1)) \cdots (X - (\alpha + p - 1))$$

in $K[X]$.

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2. Degrees of Field Extensions

Problem 4 (Fall 2009). *In $\mathbb{C}$, let $\beta = \sqrt[3]{2}$, the real cube root of 2 and let $\omega = \frac{1}{2}(-1 + \sqrt{-3})$. Set $\alpha = \beta\omega$.*
(a) Prove that $\beta + \alpha$ has degree 3 over $\mathbb{Q}$.
(b) Prove that $\beta - \alpha$ has degree 6 over $\mathbb{Q}$.

Date: November 14, 2020.

Problem 5 (Fall 2011). *Let K be a field and let $E = K(\alpha)$ be an extension field of degree $[E : K] = 37$. Prove that $K(\alpha^3)$ is also equal to E.*

Problem 6 (Spring 2013, Fall 2019). *Let K be a field and let L be an extension field of K. Let $u \in L$ and assume that the minimal polynomial of u over K is $x^n - a$ for some $a \in K$. Let $n = md$ for positive integers m, d.*

(a) Show that $[K(u^m) : K] = d$.
(b) What is the minimal polynomial of u^m over K?

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3. Fields and Galois Theory

Problem 7 (Fall 2009). *Let K be a field. Show that every automorphism ϕ of $K(t)$, the rational functions over K, fixing all elements of K is of the form*

$$\phi : t \mapsto \frac{at + b}{ct + d}$$

where $a, b, c, d \in K$ and $ad - bc \neq 0$.

Problem 8 (Spring 2012). *Suppose E/K is a Galois extension with abelian Galois group. Prove that all fields intermediate between E and K are Galois extensions of K.*

Problem 9 (Spring 2014). *Suppose that $f \in \mathbb{Q}[x]$ is an irreducible polynomial and that $\alpha, \beta \in \mathbb{C}$ are roots of f. Suppose that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ is such that $K/\mathbb{Q}$ is a finite Galois extension. Show that $\mathbb{Q}(\alpha) \cap K$ is isomorphic to $\mathbb{Q}(\beta) \cap K$.*

Hint: We know there is an isomorphism $\sigma : \mathbb{Q}(\alpha) \to \mathbb{Q}(\beta)$ sending α to β. Show that σ map extends to an automorphism of some larger field that sends K to K.

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4. Constructing Examples of Galois Field Extensions

Problem 10 (Fall 2014). *Let $K = \mathbb{Q}(\frac{-1+\sqrt{-3}}{2})$. Give an example of two non-isomorphic fields extensions L_1 and L_2 of K such that $\mathrm{Gal}(L_1/K) \cong \mathrm{Gal}(L_2/K) = \mathbb{Z}/3\mathbb{Z}$. Justify your claim.*

Problem 11 (Fall 2015). *Construct an extension field K of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois and the Galois group of K over $\mathbb{Q}$ is cyclic of order 5.*

Problem 12 (Fall 2017). *Prove the existence of an extension K of $\mathbb{Q}$ such that K is a Galois extension of $\mathbb{Q}$ with Galois group the cyclic group of order 7.*

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5. ⋆ Computing Galois Groups and Intermediate Fields ⋆

Problem 13 (Fall 2008, Spring 2018). *Let $K = \mathbb{Q}(\alpha)$ where $\alpha \in \mathbb{C}$ is a root of $f(x) = x^6 + 3$.*
(a) Show that K contains the primitie sixth roots of unity, $(1 \pm \sqrt{-3})/2$.
(b) Prove that K is Galois over $\mathbb{Q}$ and determine its Galois group.
(c) Give explicit generators for each of the intermediate fields $F, \mathbb{Q} \subseteq F \subseteq K$.

Problem 14 (Spring 2009). *Let K be the splitting field of $x^{33} - 1$ over $\mathbb{Q}$, the field of rational numbers. Determine all subfields of K (including K and $\mathbb{Q}$), and make a diagram showing all inclusions among them.*

Problem 15 (Spring 2010). *Determine the splitting field for $f(x) = (x^2 - 5)(x^2 + 2x + 2)$ over the rational field $\mathbb{Q}$. Describe explicitly the elements in its Galois group and list the subgroups and the corresponding intermediate fields.*

Problem 16 (Spring 2010). *(a) Let K be the splitting field of $x^{48} - 1$ over $\mathbb{F}_9$, the field with 9 elements. Determine the cardinality of K and make a diagram showing all subfields of K and the inclusion between them.*
(b) How many roots does $(x^2 - 5)(x^3 - 7)$ have in K?

Problem 17 (Spring 2011). *The polynomial $X^3 - 21X + 35$ is irreducible in $\mathbb{Q}[X]$.*
(a) Let α be a root (in some extension). Show that $\alpha^2 + 2\alpha - 14$ is another root.
(b) Prove that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ with Galois group cyclic of order 3.

Problem 18 (Fall 2012, Spring 2019). *Let F be the splitting field of $f = x^4 - 11$ over $\mathbb{Q}$. Show that $G = \mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to D_4, the dihedral group of order $8 = 4 \cdot 2$.*

Problem 19 (Fall 2013). *Suppose that $\alpha \in \mathbb{C}$ with $\alpha^n \in \mathbb{Q}$ such that $\mathbb{Q}(\alpha) \supseteq \mathbb{Q}$ is Galois. Further suppose that F is the field containing $\mathbb{Q}$ generated by all the roots of unity in $\mathbb{Q}(\alpha)$. Show that $\mathrm{Gal}(\mathbb{Q}(\alpha)/F)$ is a cyclic group.*

Problem 20 (Spring 2013). *Let E be a splitting field of $x^{35}-1$ over $\mathbb{F}_8$. Determine the cardinality of E and make a diagram showing all subfields of E and the inclusions between them.*

Problem 21 (Spring 2015). *Let $f(x)=x^8-1$. Find the Galois group of $f(x)$ over each of the following fields:*
(a) The rational field $\mathbb{Q}$.
(b) The field $\mathbb{Q}(i)$.
(c) The field $\mathbb{F}_3$ of three elements.

Problem 22 (Spring 2017, Fall 2018). *Let $\zeta=e^{\pi i/6}$ where $i=\sqrt{-1}$.*
(a) Find the minimal (monic) polynomial of ζ over $\mathbb{Q}$
(b) Find the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
(c) Write each subfield of $\mathbb{Q}(\zeta)$ in the form $\mathbb{Q}(u)$ (for an explicit element u).

Problem 23 (Fall 2020). *Let $\omega=e^{2\pi i/18}$ be a primitive 18-th root of unity*
1. *Find the minimal polynomial of ω over $\mathbb{Q}$.*
2. *Compute the Galois group of $\mathbb{Q}(\omega)/\mathbb{Q}$. In particular, determine if the Galois group is cyclic and give generator(s) for it.*
