ABSTRACT ALGEBRA QUALIFYING EXAM PROBLEM SESSION: MODULES

CALEB SPRINGER

Let me know if you find typos or have questions!
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Unless noted otherwise, the questions come from:
The Pennsylvania State University's Abstract Algebra Qualifying Exam (2008-2020).
`https://math.psu.edu/graduate/qualifying-exams`

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1. Decomposition into cyclic modules

Problem 1 (Fall 2010). *Consider the ring $R = \mathbb{Z}[X]$.*

(a) Is R a principal ideal domain? (Prove your answer.)
(b) Find a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ has four elements.
(c) Show that $M = \mathbb{Z}[X]/(X^2 - X, 4X + 2)$ is a finitely generated abelian group. Decompose M into a produce of cyclic groups. What is the order of $|M|$?

Hint: Determine the least $n \in \mathbb{Z} \cap (X^2 - X, 4X + 2)$ and the "least" nonconstant polynomial f not in $(X^2 - X, 4X + 2)$. Then show that the classes of 1 and f generate M.

Problem 2 (Fall 2012, Fall 2017). *Let $R = \mathbb{F}_2[t]$ and M be an R-module generated by elements a, b, c subject to the relations*

$$a + tb + (t^2 + t + 1)c = 0$$
$$(t + 1)b + (t^2 + t)c = 0.$$

Write M as a direct sum of cyclic R-modules.

Problem 3 (Fall 2019). *Let $R = \mathbb{Z}[i]$ denote the ring of Gaussian integers, i.e. $i = \sqrt{-1}$. Let $M \cong R^3$ be the free R-module of rank 3. Let $N \subseteq M$ be the*

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submodule generated by $(1, i, 4)$ and $(2, 3+2i, 14)$. Express M/N explicitly as a direct sum of cyclic R-modules.

Problem 4 (Fall 2020). *Let M be a free abelian group of rank n. Let $N \subseteq M$ be a subgroup of the same rank. Choose a basis $\{x_1, \ldots, x_n\}$ of M and a basis $\{y_1, \ldots, y_n\}$ of N, and expand*

$$y_i = \sum_{j=1}^{n} a_{ij} x_j,\ 1 \le i \le r,\ a_{ij} \in \mathbb{Z}.$$

Let A be the $n \times n$ matrix (a_{ij}) whose i-th row consists of the coefficients in the expansions of y_i. Prove that M/N is a finite group of order $|\det(A)|$.

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2. Isomorphism Problems

Problem 5 (Fall 2013). *Let R denote the ring $\mathbb{Q}[x]$ and let N denote the R-module $R/(x^2+1)$. Further suppose that M and M' are finitely generated R-modules such that*

$$M \oplus N \cong M' \oplus N,$$

in other words $M \oplus N$ and $M' \oplus N$ are isomorphic as R-modules. Prove that $M \cong M'$ as R-modules.

Problem 6 (Spring 2014). *Suppose that $R = \mathbb{Z}[i]$ and that M and N are finitely generated R-modules. Suppose further that $P = (1+i)$.*

1. *Show that P is a prime ideal.*
2. *Suppose that $M \oplus R/P \oplus P$ is isomorphic to $N \oplus R/P \oplus P$. Prove that M and N are isomorphic.*

Problem 7 (Fall 2018). *Let R be a principal ideal domain. Let a and b be elements of R. Prove that the R-module of R-module homomorphisms from R/aR to R/bR is 0 if $a \neq 0$ and $b = 0$. Prove that it is isomorphic to $R/\gcd(a,b)R$ when a and b are both nonzero. In each case, describe explicitly a homomorphism that generates this cyclic module by specifying its value on the image of 1 in R/aR.*

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3. Modules over a Polynomial Ring

Problem 8 (Fall 2015). *An R-module M is called* irreducible *if $M \neq 0$ and if 0 and M are the only R-submodules of M. Let $R = \mathbb{Q}[x]$. Construct two non-isomorphic irreducible R-modules whose underlying abelian group is $\mathbb{Q} \times \mathbb{Q}$.*

Problem 9 (Fall 2016, Spring 2019). *Let $\mathbb{F}_2$ be a field with 2 elements and let $R = \mathbb{F}_2[X]$. List up to isomorphism all R-modules with 8 elements that are cyclic. Justify your answer. Give an example of an R-module with 8 elements which is not cyclic.*

Problem 10 (Spring 2018). *Let R be a PID. Let us consider the quotient $M = R[t]/t^2R[t]$ as a module over the polynomial ring $R[t]$. Prove that M is* not *isomorphic to a direct sum $M_1 \oplus M_2$ of two nonzero $R[t]$ modules M_1 and M_2.*

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4. Free and Torsion

Problem 11 (Spring 2015). *An element m of an R-module M is called a* torsion element *if $rm = 0$ for some nonzero element $r \in R$. Let R be a principal ideal domain. Let F be the fraction field of R. Let R be a principal ideal domain. Let F be the fraction field of R. Let M_1 and M_2 be finitely generated R-modules such that*

$$M_1 \otimes_R F \cong M_2 \otimes_R F$$

(a) Prove that if both M_1 and M_2 have no torsion elements then $M_1 \cong M_2$.
(b) Give an explicit example which shows that the conclusion in (a) is false if M_1 or M_2 have torsion elements.

Problem 12 (Spring 2017). *Let F be a field and R be the ring of all polynomial $f(x) \in F[x]$ with $f'(0) = 0$. Give an example of a non-free submodule N of a free R-module M.*
