ABSTRACT ALGEBRA QUALIFYING EXAM PROBLEM SESSION: RINGS AND IDEALS

CALEB SPRINGER

Let me know if you find typos or have questions!
Email: `cks5320@psu.edu`

Unless noted otherwise, the questions come from:
The Pennsylvania State University's Abstract Algebra Qualifying Exam (2008-2020).
`https://math.psu.edu/graduate/qualifying-exams`

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1. Irreducible Polynomials

Problem 1 (Spring 2011). *(a) Let $K = \mathbb{Z}/2\mathbb{Z}$. The only irreducible polynomials in $K[X]$ of degree ≤ 2 are X, $X+1$ and X^2+X+1. Using this, show that X^5+X^3+1 is irreducible in $K[X]$.*
(b) Let r and s be odd integers. Prove that X^5+rX^3+s is always irreducible in $\mathbb{Q}]X]$. You may use Gauss' Lemma.

Problem 2 (Fall 2016). *Let F be a field of prime characteristic p such that the polynomial $X^p - X - 1$ is reducible in $F[X]$. Show that it splits into a product of p distinct monic linear factors in $F[X]$. Justify your answers.*

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2. Describing and Counting (Prime/Maximal) Ideals

Problem 3 (Fall 2008). *Let $R = \mathbb{Z}[x]$ be the ring of polynomials in one variable over the integers. For a prime number p, let I_p denote the ideal of R generated by p and x^2+1. For each of $p = 2, 3, 5$, determines whether I_p is prime and determine all the maximal ideals of R containing I_p.*

Problem 4 (Spring 2009). *Let $R = \mathbb{Z}[i]$. For each of the following ideals in $R[x, y]$, decide whether it is a prime ideal, giving your justification. If it is not maximal, give a maximal ideal containing it.*

Date: November 2, 2020.

(a) (x, y)
(b) $(x, 2y)$
(c) $(x^2 + 1, y)$
(d) $(3, x^2 + 1, y)$
(e) $(5, x^2 + 1, y)$

Problem 5 (Spring 2010, Fall 2019)**.** *Determine the maximal ideals* $\mathfrak{m}$ *of the polynomial ring* $\mathbb{Z}[x]$ *in one variable over the integers that contain the integer* 30 *and the polynomial* $x^2 + 1$*. Give explicitly two generators for each such maximal ideal* $\mathfrak{m}$*. How many such maximal ideals are there?*

Problem 6 (Fall 2012)**.** *Let* $R = \mathbb{Z}[x]$ *be the ring of polynomials with integer coefficients and* $I = (3, x^2+3x+5)$ *the ideal in* R *generated by* 3 *and* x^2+3x+5*. Find all prime ideals of* R *containing* I*.*

Problem 7 (Fall 2013)**.** *Find an ideal* I *in the ring* $A = \mathbb{Z}[x]$ *such that* A/I *has exactly three prime ideals. Identify the ideals and justify your assertion.*

Problem 8 (Fall 2014)**.** *Let* $F[x]$ *be the ring of polynomials with coefficients in an algebraically closed field* F*. Describe, with justification,*

(1) All ideals of $F[x] \times F[x]$*.*
(2) All prime ideals of $F[x] \times F[x]$*.*

Problem 9 (Spring 2014)**.** *Find all prime ideals in* $R = \mathbb{Z}[i][x]/(1+i, x^2+2)$*.*

Problem 10 (Fall 2015)**.** *Count the number of prime ideals in the ring*

$$\mathbb{Z}[x, y]/(6, (x-2)^2, y^6).$$

How many of these prime ideals are maximal? (The whole ring is not considered a prime ideal.)

Problem 11 (Fall 2017)**.** *Let* R *be the ring* $\mathbb{Z}[x]/(x^2 - 2)$*. Which of the following ideals are prime:* $(0), (2), (3), (7)$*? Which are maximal?*

Problem 12 (University of Michigan, August 2020)**.** *Let* I *be the ideal* $\big(x^2 + 1, y^2 + 1\big)$ *in the ring* $\mathbb{Q}[x, y]$*. Show that* I *is not prime and give a prime ideal containing* I*. (We remind the reader that the ideal* (1) *is not considered to be prime.)*

3. Integral Domains

Problem 13 (Spring 2012)**.** *Suppose that an integral domain D has a field F inside it, and D is a finite-dimensional vector space over F. Prove that D is actually a field itself.*

Problem 14 (Fall 2018)**.** *Let us consider the ring*

$$\mathcal{O} = \{a + 2b\sqrt{6} \mid a, b \in \mathbb{Z}\}.$$

Prove that the multiplicative group $\mathcal{O}^*$ of invertible elements of $\mathcal{O}$ is infinite.

3.1. **Principal Ideal Domains.**

Problem 15 (Fall 2010)**.** *Consider the ring $R = \mathbb{Z}[X]$.*

(a) Is R a principal ideal domain? (Prove your answer.)
(b) Find a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ has four elements.
(c) Show that $M = \mathbb{Z}[X]/(X^2 - X, 4X + 2)$ is a finitely generated abelian group. Decompose M into a produce of cyclic groups. What is the order of $|M|$?

Hint: Determine the least $n \in \mathbb{Z} \cap (X^2 - X, 4X + 2)$ and the "least" nonconstant polynomial f not in $(X^2 - X, 4X + 2)$. Then show that the classes of 1 and f generate M.

Problem 16 (Spring 2013)**.** *Let R be a principal ideal domain and let I, J be two nonzero ideals of R. Show that $IJ = I \cap J$ if and only if $I + J = R$.*

Problem 17 (Spring 2015)**.** *Let $R = \mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$.*

(a) Compute the factor rings $R/(2)$ and $R/(2i)$. Are these rings isomorphic?
(b) Is R a principal ideal domain?

Justify your answers.

Problem 18 (Fall 2016)**.** *Determine whether $\mathbb{Z}[\sqrt{-6}]$ is a principal ideal domain. Justify your answer.*

Problem 19 (Spring 2019)**.** *Let R be an integral domain. Let $a \in R$ be a nonzero element which is not a unit. We say that a is* irreducible *if $a = bc$ where $b, c \in R$ implies that either b or c is a unit. We say that a is* prime *if $ad = bc$ where $b, c, d \in R$ implies that either $b = aa'$ or $c = aa'$ for some $a' \in R$.*

Prove that if R is a principal ideal domain, then any irreducible element is prime.

3.2. **Unique Factorization Domains.**

Problem 20 (Spring 2017)**.** *Suppose that S is a unique factorization domain, and R is a subring of S with the following property: If $f \in S$ and $g \in R$ such that f divides g, then $f \in R$. Show that R must also be a unique factorization domain.*

Hint: Show first that both rings must have the same units, i.e. that $R^* = S^*$.

Problem 21 (Spring 2018)**.** *Let R be a UFD, and let f be any nonzero, non-unit irreducible element of R. Prove that $R/(f)$ is also a UFD or disprove by counterexample. What happens if R is also a PID?*

3.3. **Euclidean Domains.**

Problem 22 (Fall 2009)**.** *We say that a ring R is a* Euclidean Domain *if there is a function $\delta : R \to \mathbb{Z}_{\geq 0}$ such that for each pair of elements a, b from R with b non-zero, there exists q and r in R such that $a = bq + r$ and $\delta(r) < \delta(b)$. Show that $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ is a Euclidean domain.*

Problem 23 (Fall 2020)**.** *Let R be the subring of $\mathbb{Q}[x]$ consisting of those polynomials $a_0 + a_2x^2 + \cdots + a_nx^n$ whose coefficient of x is 0. Show that x^2 is irreducible in R, but x^2 is not prime. Is R a Euclidean domain?*