ABSTRACT ALGEBRA QUALIFYING EXAM PROBLEM SESSION: GROUPS AND GROUP ACTIONS

CALEB SPRINGER

Let me know if you find typos or have questions!
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Unless noted otherwise, the questions come from:
The Pennsylvania State University's Abstract Algebra Qualifying Exam (2008-2020).
`https://math.psu.edu/graduate/qualifying-exams`

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1. Generators and relations

Problem 1 (Spring 2011)**.** *Suppose we have an abelian group given by generators x, y, z and relations*

$$\begin{aligned} 2x + 6y + 4z &= 0 \\ 4x + 2y + 10z &= 0 \\ 6x - 2y + 16z &= 0 \end{aligned}$$

Find a product of cyclic groups isomorphic to this group.

Problem 2 (Spring 2012)**.** *Suppose we have an abelian group given by generators x, y, z and relations*

$$\begin{aligned} 3x + 2y + 4z &= 0 \\ 2x - y + 10z &= 0 \\ 5x + 3y - 2z &= 0 \end{aligned}$$

Find a product of cyclic groups isomorphic to this group.

Date: October 26, 2020.

Problem 3 (Fall 2014). *Let M be the cokernel of the mapping from $\mathbb{Z}^2$ to $\mathbb{Z}^3$ given by the matrix*

$$\begin{bmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{bmatrix}.$$

How many $\mathbb{Z}$-module homomorphisms are there from M to $\mathbb{Z}/3\mathbb{Z}$?

Problem 4 (Fall 2020). *Let M be a free abelian group of rank n. Let $N \subseteq M$ be a subgroup of the same rank. Choose a basis $\{x_1, \ldots, x_n\}$ of M and a basis $\{y_1, \ldots, y_n\}$ of N, and expand*

$$y_i = \sum_{j=1}^{n} a_{ij} x_j,\ 1 \leq i \leq r,\ a_{ij} \in \mathbb{Z}.$$

Let A be the $n \times n$ matrix (a_{ij}) whose i-th row consists of the coefficients in the expansions of y_i. Prove that M/N is a finite group of order $|\det(A)|$.

1.1. **Check your answers:** Some of the problems in this section have required calculating the *Smith Normal Form* of a matrix with integer coefficients. You can easily check your answers with the SageMathCell calculator `https://sagecell.sagemath.org/`. Sage is an open-source option for computations regarding abstract algebra, number theory, and other subjects. It is based on Python, and this webpage is useful for small computations, like the one below.

Example: Type the following into the SageMathCell calculator.

```
M = Matrix( [[1,2,3],[4,5,6],[7,8,9]])
M.smith_form()
```

This computes that the Smith Normal Form of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(There will be two other matrices listed, and these are simply the change of basis matrices U and V so that the Smith Normal Form is equal to UMV.)

Warning: Ultimately, you need to be able to do all computations by hand on the qualifying exam! Make sure you don't get too reliant on calculators. I just recommend the calculator as a way of checking yourself.

2. Simplicity

Problem 5 (Fall 2008, Spring 2017)**.** *No group of order* 72 *is simple.*

Problem 6 (Spring 2009)**.** *Show that no group of order* 392 *can be simple.*

Problem 7 (Fall 2010)**.** *Show that no group of order* $10,000$ *can be simple.*

Problem 8 (Fall 2013)**.** *Show there is no simple group of order* 108*.*

Problem 9 (Spring 2013)**.** *Show that a group of order* 36 *cannot be simple,*

Problem 10 (Spring 2014)**.** *There is no simple group of order* 96*.*

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3. Classification of Groups

Problem 11 (Spring 2011)**.** *Let G be a finite group of order* $91 = 7 \cdot 13$*. Prove that G is cyclic.*

Problem 12 (Fall 2015, Spring 2019)**.** *Let G be a group of order* 30*. Show that G has a normal subgroup that is cyclic of order* 15*.*

Problem 13 (Fall 2011)**.** *Determine the number of nonisomorphic commutative groups of order* $360 = 2^3 3^2 5$ *exist, and write each one as a product of cyclic groups of prime power orders.*

Problem 14 (Spring 2015)**.** *Let p be a prime number. Classify, up to isomorphism, all group of order* $2p$*.*

Problem 15 (Fall 2016)**.** *Determine, up to isomorphism, all groups of order* 35*.*

Problem 16 (Fall 2020)**.** *List all finite groups which have the property: If $g, h \in G$, then either g is a power of h, or h is a power of g.*

Problem 17 (University of Michigan, August 2020)**.** *Counting up to isomorphism, how many abelian groups G are there such that G is generated by* 3 *elements and $g^4 = 1$ for all $g \in G$?*

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4. Group Actions

Problem 18 (Spring 2010, Fall 2011). *Let H be a group of order $5^r 7^s$ for integers $r \geq 0$, $s \geq 0$ and suppose that H acts on a set X with 23 elements. Prove that some element of X is fixed by the action.*

Problem 19 (Fall 2012). *Show that the alternating group A_6 has no subgroup of order $72 = 6 \cdot 4 \cdot 3$. You may use the fact that A_6 is simple.*

Problem 20 (Fall 2014). *Let G be a finite group of order p^n, where p is prime and $n \geq 1$. Suppose G acts on a finite set S. Let S' be the subset of S consisting of elements fixed by G:*

$$S' = \{x \in S : gx = x \,\forall g \in G\}.$$

Prove that the order of S' is congruent to the order of S modulo p.

Problem 21 (Fall 2017). *Let S be a set of all elements of order 7 in the alternating group A_7. Prove that S is* not *a conjugacy class in A_7.*

Problem 22 (Spring 2018). *A group G is said to act faithfully on a set X of cardinality n if the corresponding homomorphism $\phi : G \to S_n$ is injective. Show that an abelian group of order 100 cannot act faithfully on a set with 13 elements. (Hint: Consider the 5-Sylow subgroup H of G and determine its centralizer in S_{13}.)*

Problem 23 (Stanford, Fall 2009). *Let G be a nontrivial finite group and p be the smallest prime dividing the order of G. Let H be a subgroup of index p. Show that H is normal. (Hint: If H isn't normal, consider the action of G on the conjugates of H.)*

Problem 24 (Stanford, Spring 2011). *(a) Prove that if G is a finite group and H is a proper subgroup, then G is not a union of conjugates of H. (Hint: The conjugates all contain the identity.)*

(b) Suppose G is a finite transitive group of permutations of a finite set X of n objects, $n > 1$. Prove that there exists $g \in G$ with no fixed points in X. (Hint: Use part (a).)

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5. Subgroups (Miscellaneous)

Problem 25 (Fall 2010). *Let S_7 be the group of all permutations of the set $\{1, 2, 3, 4, 5, 6, 7\}$. Let $g = (123)(4567) \in S_7$ be the product of a 3-cycle and a 4-cycle that are disjoint. Let G denote the subgroup of S_7 generated by g.*

(a) What is the order of G?
(b) How many conjugates does g have in S_7?
(c) What is the order of the centralizer of g in S_7?
(d) How many distinct conjugates hGh^{-1}, $h \in S_7$ does G have?

Problem 26 (Spring 2012)**.** *Let S_i be a family of subgroups in some group G. Suppose that an element $x_i \in G$ is chosen for each i, and we form the intersection of all the cosets $x_i S_i$. Prove that the intersection is either empty or a coset of the intersection of all the S_i.*

Problem 27 (Fall 2018)**.** *Let G and H be finite groups. Show that every p-Sylow subgroup of $G \times H$ is of the form $P \times Q$ for P a p-Sylow subgroup of G and Q and p-Sylow subgroup of H.*

Problem 28 (Fall 2019)**.** *Let H be a subgroup of a group G of index 3. Show that either H is normal or H has a subgroup N of index 2 in H such that N is normal in G.*

Problem 29 (Stanford, Fall 2014)**.** *Suppose G is a group of odd order and p is the smallest prime dividing $|G|$. If the p-Sylow subgroup $S \subseteq G$ is normal and has order p^2 or p, prove that S is contained in the center of G.*

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6. Solvable Groups

Problem 30 (Fall 2009)**.** *Every finite group contains a unique maximal normal solvable subgroup.*

Problem 31 (Spring 2009)**.** *Prove that if H and K are solvable subgroups of G and K is normal in G, then HK is solvable in G.*